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Action of Amenable Groups and Uniqueness of Invariant Means

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It is proved in this paper that assuming the continuum hypothesis, there exists an amenable group G acting on an infinite set X such that there is only one G-invariant mean on $l^\infty(X)$. © 1991 Academic Press, Inc.

1. INTRODUCTION

Let G be a group acting on an infinite set X. Then G acts on the Banach space $l^\infty(X)$ of all bounded real functions on X by ${}_gf(x)=f(gx)$, for any f in $l^\infty(X)$, any g in G, and any x in X. A mean μ on $l^\infty(X)$ is a positive linear functional on $l^\infty(X)$ such that $\|\mu\|=1$. μ is a G-invariant mean if $\mu({}_gf)=\mu(f)$ for all $f\in l^\infty(X)$ and all $g\in G$. When G acts on itself by left multiplication, a G-invariant mean on $l^\infty(G)$ is called a left invariant mean. G is said to be amenable if there is a left invariant mean on $l^\infty(G)$ (see [11]).

The study of the existence and uniqueness of invariant means was initiated in Banach [1] and Banach and Tarski [2], where they showed that there exists more than one mean on $\mathbb{R}$ and $\mathbb{R}^2$, but none on $\mathbb{R}^3$, invariant under the groups of all translations and rotations. Von Neumann [21] noted that the difference comes from the structure of the transformation groups. He proved that G-invariant means exist if G is amenable. On the uniqueness problem, after Day's work in [6], Granirer [9] proved that any infinite amenable group G admits many left invariant means. Later on, Rosenblatt and Talagrand in [19] and Paterson in [16] showed that if G is amenable, X is infinite and $|G|\leqslant|X|$, then there are $2^{2^{|X|}}$ G-invariant means. But the general problem still remains open.

PROBLEM. If G is an amenable group acting on an infinite set X, can there exist exactly one G-invariant mean on $l^\infty(X)$?

Comments on this problem can be found in [13, 16, 18, 19].

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In this paper we are going to prove that under the continuum hypothesis (CH), there exists a locally finite, hence amenable, group G with $|G| = c$ acting on a countable infinite set X, such that there is only one G-invariant mean on $l^\infty(X)$. Also some related examples are given. One of them is a nilpotent group G which has an abelian subgroup H with $|H| = 2^{[G:H]}$, such that H does not contain any nontrivial normal subgroup of G.

For $H < G$, $x \in X$, and $A \subset X$, we denote by $Hx = \{gx \mid g \in H\}$ and $HA = \bigcup \{Hx \mid x \in A\}$.

If μ is a mean on $l^\infty(X)$ and $g \in G$, we define $g \cdot \mu \in l^\infty(X)^*$ by $g \cdot \mu(f) = \mu({}_g f)$ for $f \in l^\infty(X)$. $g \cdot \mu$ is also a mean, and μ is G-invariant if and only if $g \cdot \mu = \mu$ for all $g \in G$. A mean on $l^\infty(X)$ which is an element of $l^1(X)$ is called a countable mean, and a countable mean with a finite support is a finite mean.

For each $x \in X$, we define an operator T_x from $l^1(G)$ to $l^1(X)$ as $\forall f \in l^1(G)$, $\forall y \in X$, let

$$T_x f(y) = \sum_{g(x) = y} f(g).$$

This operator is linear and of norm 1. Thus it maps a (countable) mean on $l^\infty(G)$ into a mean on $l^\infty(X)$.

Let r be a real number with $0 \leqslant r \leqslant 1$. A subset A of X is said to be $G(r)$-thick if for any finite subset F of G, there exists $x \in X$, such that

$$|\{g \in F \mid gx \in A\}| \geqslant r\,|F|.$$

When A is $G(1)$-thick, we simply say A is G-thick. This concept is a quantitative generalization of left thick sets in Mitchell [14]. The following proposition partially generalizes [10, Proposition 3].

PROPOSITION 1. *If G is amenable, a set $A \subset X$ is $G(r)$-thick if and only if there exists a G-invariant mean μ on $l^\infty(X)$ such that $\mu(A) \geqslant r$.*

Proof. Suppose A is $G(r)$-thick. Let Λ be the set of all finite subsets of G. Since G is amenable, it satisfies the Følner condition (see [8] or Namioka [15]). Thus for each $\lambda \in \Lambda$, there is a finite subset F_λ of G, such that $\forall g \in \lambda$,

$$|F_\lambda \backslash gF_\lambda| < \frac{1}{2\,|\lambda|}\,|F_\lambda|.$$

Let μ_λ be the arithmetic average on F_λ. Then μ_λ is a finite mean on $l^\infty(G)$, and $\forall g \in \lambda$, $\|\mu_\lambda - g \cdot \mu_\lambda\| < 1/|\lambda|$. Therefore, for any $x \in X$ and $g \in \lambda$,

$$\begin{aligned} \| g \cdot (T_x \mu_\lambda) - T_x \mu_\lambda \| &= \| T_x(g \cdot \mu_\lambda) - T_x \mu_\lambda \| \\ &= \| T_x(g \cdot \mu_\lambda - \mu_\lambda) \| < 1/|\lambda|, \end{aligned}$$

since for any $y \in X$,

$$T_x(g \cdot \mu_\lambda)(y) = \sum_{h(x)=y} g \cdot \mu_\lambda(h) = \sum_{h(x)=y} \mu_\lambda(g^{-1}h)$$
$$= \sum_{h(x)=g^{-1}(y)} \mu_\lambda(h) = T_x \mu_\lambda(g^{-1}(y)) = g \cdot (T_x \mu_\lambda)(y).$$

Since A is $G(r)$-thick, there exists $x_\lambda \in X$ such that

$$|\{g \in F_\lambda \mid gx_\lambda \in A\}| \geqslant r\, |F_\lambda|.$$

Thus we have $T_{x_\lambda}\mu_\lambda(A) \geqslant r$. Note that Λ is a directed set directed by inclusion. So we obtain a net $\{T_{x_\lambda}\mu_\lambda\}$ of finite means on $l^\infty(X)$, satisfying $\| g \cdot (T_{x_\lambda}\mu_\lambda) - T_{x_\lambda}\mu_\lambda \| \to 0$ for any $g \in G$. Take μ to be any w^*-limit point of the net $\{T_{x_\lambda}\mu_\lambda\}$ in $l^\infty(X)^*$. Then μ is a G-invariant mean and $\mu(A) \geqslant r$.

Conversely, suppose A is not $G(r)$-thick. Then there is a finite subset F of G such that for any $x \in X$,

$$|\{g \in F \mid gx \in A\}| < r\, |F|.$$

Let μ be any G-invariant mean on $l^\infty(X)$. Then

$$\mu(A) = \frac{1}{|F|} \sum_{g \in F} \mu({}_g\chi_A) = \mu\left(\frac{1}{|F|} \sum_{g \in F} \chi_{g^{-1}A}\right) < r. \quad \blacksquare$$

The same proof, with the use of the Følner condition, is not valid for the semigroup case (cf. [12, 22]). But it works when $r = 1$.

2. Main Theorem

When G acts on itself by left multiplication, there are $|G|$ many disjoint left thick subsets in G if G is infinite and amenable (Chou [5]). In general, if X is infinite, G is amenable and $|G| \leqslant |X|$, this is also true as shown in [16, 19]: There are $|X|$ many disjoint G-thick subsets in X. When $|G| > |X|$, it is no longer true as we will show. The next two lemmas are simple algebraic facts which are very important in our argument.

Lemma 1. *Let G be a finite group acting on an infinite set X, and let $\mathscr{A}$ be a decomposition of X by finite G-invariant subsets. Suppose there is a number n such that $|A| < n$ for each $A \in \mathscr{A}$. Let g be a permutation of X that leaves each $A \in \mathscr{A}$ invariant. Then the group generated by G and g is also finite.*

Proof. G has only finitely many different representations on sets of cardinality $< n$ (up to equivalence). So there are only finitely many different combinations of G and g on the sets $A \in \mathscr{A}$. This means that the new group has a faithful representation on a finite set. Thus it is finite. ∎

LEMMA 2. *With the conditions of Lemma 1, and if $\mathscr{A}$ is a cover instead of a decomposition, then the new group generated by G and g is still finite.*

Proof. By taking finite intersections of all the sets in the cover $\mathscr{A}$ and their complements, we obtain a Boolean ring of finite subsets of X. All the minimal elements of this Boolean ring form a decomposition of X satisfying the conditions of Lemma 1. ∎

A group G acting transitively on a set X means that for any $x, y \in X$, there exists $g \in G$ with $gx = y$. Note that if G is not transitive then there are two disjoint G-thick sets.

THEOREM 1. *Let G be a countable locally finite group acting transitively on a countable infinite set X. Suppose A is an infinite subset of X such that $X \backslash A$ is G-thick. Then there exists a permutation g of X such that $g(A) \subset X \backslash A$ and the group $\bar{G}$ generated by G and g is still locally finite. Furthermore, g can be chosen so that the set $X \backslash (A \cup g(A))$ is $\bar{G}$-thick.*

Proof. Write $G = \bigcup G_n$, where $G_1 < G_2 < G_3 < \cdots$ is a sequence of finite subgroups of G. Since G acts transitively on X, for any $x \in X$, $\bigcup G_i x = Gx = X$. Also for any finite subset K of X and any G-invariant mean μ on $l^\infty(X)$, $\mu(K) = 0$. Thus $X \backslash (A \cup K)$ is G-thick since $X \backslash A$ is so.

Choose $x_1 \in A$. Let $A_1 = A \cap G_1 x_1$. Write $A_1 = \{x_1, x_2, ..., x_{n_1}\}$ and choose distinct elements $y_1, y_2, ..., y_{n_1}$ in $X \backslash A$. Define $A_2 = A \cap \bigcup_{i=1}^{n_1} (G_2 x_i \cup G_2 y_i)$, and write $A_2 = \{x_1, ..., x_{n_1}, x_{n_1+1}, ..., x_{n_2}\}$. Take $y_{n_1+1} \in X$ such that $G_1 y_{n_1+1} \subset X \backslash (A \cup \{y_1, ..., y_{n_1}\})$. Then we choose y_{n_1+2} such that $G_1 y_{n_1+2} \subset X \backslash (A \cup \{y_1, ..., y_{n_1+1}\})$, and so on. After choosing y_{n_2} such that $G_1 y_{n_2} \subset X \backslash (A \cup \{y_1, ..., y_{n_2-1}\})$, we define $A_3 = A \cap \bigcup_{i=1}^{n_2}$-$(G_3 x_i \cup G_3 y_i)$ and write $A_3 = \{x_1, ..., x_{n_2}, x_{n_2+1}, ..., x_{n_3}\}$. Similarly, we choose $y_{n_2+1} \in X$ such that $G_2 y_{n_2+1} \subset X \backslash (A \cup \{y_1, ..., y_{n_2}\})$, ..., choose y_{n_3} such that $G_2 y_{n_3} \subset X \backslash (A \cup \{y_1, ..., y_{n_3-1}\})$. Continuing this procedure, we obtain a sequence $\{A_n\}$ of finite subsets of A, a sequence of integers $n_1, n_2, n_3, ...,$ and a sequence of elements $y_1, y_2, y_3, ...,$ in $X \backslash A$, satisfying

(i) $A = \bigcup A_k$ and $A_1 \subset A_2 \subset A_3 \subset \cdots$;

(ii) $|A_k| = n_k$;

(iii) $A \cap G_k A_k = A_k$;

(iv) $n > n_k \Rightarrow G_k y_n \subset X \backslash (A \cup \{y_1, ..., y_{n-1}\})$;

(v) $n \leqslant n_k \Rightarrow G_k y_n \cap A \subset A_{k+1}$.

Define g as following. If $x \in A$, then $x = x_n$ for some n. Let $g(x_n) = y_n$ and $g(y_n) = x_n$. If $x \notin A$ and $x \neq y_n$ for every n, then let $g(x) = x$.

Let $\bar{G}[\bar{G}_k]$ be the group generated by $G[G_k]$ and g. Then $\bar{G} = \bigcup \bar{G}_k$. We prove that each $\bar{G}_k$ is finite.

Take $z \in X$. If $G_k z \cap (A \cup g(A)) = \emptyset$, then $G_k z$ is invariant under both G_k and g. Suppose $x \in G_k z$ for some $x \in A$. If $x \in A_{k+1}$, then $G_k A_{k+1} \cup \bigcup_{i=1}^{n_{k+1}} G_k y_i$ is invariant under both G_k and g (by (iii), (iv), and (v)) with cardinality $\leqslant 2n_{k+1} \cdot |G_k|$. If $x \notin A_{k+1}$, then $G_k x \cap A_{k+1} = \emptyset$. So $G_k x \cup G_k(g(G_k x))$ is invariant under both g and G_k (by (iv)) with cardinality $\leqslant |G_k|(|G_k| + 1)$. Note that these invariant sets contain all the y_n's. Thus we have obtained a cover of X (in fact a decomposition) satisfying the conditions of Lemma 1. By Lemma 2, $\bar{G}_k$ is finite.

Finally, we show that the set $X \backslash (A \cup g(A))$ can be $\bar{G}$-thick if g is properly chosen. Since $X \backslash A$ is G-thick, it can be split into two G-thick subsets in the following way. Take a_1, b_1 in X such that $G_1 a_1$ and $G_1 b_1$ are disjoint and contained in $X \backslash A$. Then take a_2, b_2 in X such that $G_2 a_2$, $G_2 b_2$ are disjoint and contained in $X \backslash (A \cup G_1 a_1 \cup G_1 b_1)$. In this way we obtain disjoint subsets $G_1 a_1, G_2 a_2, G_3 a_3, \ldots$, and $G_1 b_1, G_2 b_2, G_3 b_3, \ldots$, of $X \backslash A$. The two sets $B_1 = \bigcup G_k a_k$ and $B_2 = \bigcup G_k b_k$ are disjoint G-thick subsets of $X \backslash A$. If we choose g so that $g(A) \subset B_1$, then $B_2 \subset X \backslash (A \cup g(A))$ and is $\bar{G}$-thick. ∎

COROLLARY. *Under the same conditions as in Theorem* 1, *there exists a countable locally finite group $\bar{G}$ containing G, such that for each $\bar{G}$-invariant mean μ on $l^\infty(X)$, $\mu(A) = 0$.*

Proof. Let $G_0 = G$. Inductively, we can define a sequence of countable locally finite groups $G_0 < G_1 < G_2 < \cdots$, a sequence of permutations $g_1, g_2, g_3, \ldots$, of X, and a sequence of subsets $A = A_0 \subset A_1 \subset A_2 \subset \cdots$ of X, such that G_{k+1} is generated by G_k and g_{k+1}, A_{k+1} is the disjoint union of A_k and $g_{k+1}(A_k)$, and each $X \backslash A_k$ is G_k-thick.

Let f_k be the characteristic function of A_k. Then $f_{k+1} = f_k + {}_{g_{k+1}}f_k$. Thus if μ is a G_k-invariant mean on $l^\infty(X)$, then $\mu(f_0) \leqslant 2^{-k}$. Define $\bar{G} = \bigcup G_k$. Then $\bar{G}$ has the properties we need. ∎

From this corollary, we see that we can turn a G-thick set whose complement is also G-thick into a null set under the new group $\bar{G}$. If the complement is only, say, $G(\frac{1}{2})$-thick, then we need a more delicate argument. The following is an equivalent definition for $G(r)$-thick sets when G is locally finite.

PROPOSITION 2. *Let G be a locally finite group acting on a set X and $A \subset X$. Then A is $G(r)$-thick if and only if for any finite subgroup H of G, there exists $x \in X$, such that*

$$|Hx \cap A| \geqslant r\,|Hx|.$$

Proof. If H is a subgroup, then

$$|\{g \in H \mid gx \in A\}|/|H| = |Hx \cap A|/|Hx|.$$

On the other hand, from the condition in the proposition, one can prove as in the proof of Proposition 1 that there exists a G-invariant mean μ on $l^\infty(X)$ such that $\mu(A) \geqslant r$. ▮

THEOREM 2. *Suppose G is a countable locally finite group acting transitively on an infinite set X, and $A \subset X$. If A is $G(r)$-thick for some $r > \frac{1}{2}$, then there exists a G-thick set $B \supset A$ and a permutation g of X, such that g fixes points in $X \backslash B$, $g(B \backslash A) \subset A$, and the group $\bar{G}$ generated by G and g is still locally finite.*

Proof. Write $G = \bigcup G_i$, where $\{e\} = G_0 < G_1 < G_2 < \cdots$ is an increasing sequence of finite subgroups of G. Choose a nonincreasing sequence of positive real numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$, such that $\sum \varepsilon_i < r - \frac{1}{2}$. By taking a subsequence of G_n, we may suppose that $\varepsilon_n |G_{n+1}| > 3 |G_n|$ for all $n \geqslant 1$.

Choose a sequence of finite mutually disjoint subsets of X: $B_1, B_2, B_3, \ldots$, such that each B_i is invariant under G_i, $|B_i \cap A| \geqslant r |B_i|$, and $|G_i| \leqslant |B_i| < 3 |G_i|$. This is possible since, by Propositions 1 and 2, for any finite subset K of A, $A \backslash K$ is also $G(r)$-thick by the transitivity of G, and for any minimal G_i-invariant subset $G_i x$ of X, $|G_i x| \leqslant |G_i|$. The set $\bigcup B_i$ is G-thick, hence so is $B = A \cup \bigcup B_i$.

Now fix a B_n. We divide B_n into some disjoint G_{n-1}-invariant subsets $B_n(1)$, $B_n(2), \ldots$, such that $|G_{n-1}| \leqslant |B_n(i)| < 3 |G_{n-1}|$. For each $B_n(i)$, we also divide it into disjoint G_{n-2}-invariant subsets $B_n(i, 1)$, $B_n(i, 2), \ldots$, such that $|G_{n-2}| \leqslant |B_n(i, j)| < 3 |G_{n-2}|$. After $n-1$ steps we reach some G_1-invariant sets $B_n(k_{n-1}, k_{n-2}, \ldots, k_2, k_1)$, satisfying

(i) $B_n(k_{n-1}, k_{n-2}, \ldots, k_2, k_1) \subset B_n(k_{n-1}, k_{n-2}, \ldots, k_2) \subset \cdots \subset B_n(k_{n-1}, k_{n-2}) \subset B_n(k_{n-1})$, and

(ii) $|G_i| \leqslant |B_n(k_{n-1}, \ldots, k_i)| < 3 |G_i|$.

Denote each $x \in B_n$ by $x = x(k_{n-1}, \ldots, k_1, k_0)$ such that $x(k_{n-1}, \ldots, k_1, k_0) \in B_n(k_{n-1}, \ldots, k_1)$, and k_0 runs through $1, 2, \ldots, |B_n(k_{n-1}, \ldots, k_1)|$. Let the elements of B_n be in the lexicographical order; i.e., the first element is $x(1, \ldots, 1, 1)$ and the second is $x(1, \ldots, 1, 2)$, and so on. Each set $B_n(k_{n-1}, \ldots, k_i)$ is called an (n, i)-segment. For each $(n, i+1)$-segment K and any (n, i)-segment K', we have $\varepsilon_i |K| > |K'|$, whether or not $K' \subset K$. Also we note that the only (n, n)-segment is B_n itself.

Now we are going to define a $1-1$ maping g from $B_n \backslash A$ into $B_n \cap A$ in stages. This is possible since $|B_n \cap A| \geqslant r |B_n| > |B_n|/2$. Let $C_{-1} = B_n \backslash A$, $C_0 = C_1 = \cdots = C_{n-2} = \varnothing$ for the moment. The sets C_i, $i = 0, 1, \ldots, n-2$, are subject to change in the following procedure. In this procedure C_0 is defined as the image of g, thus $C_0 \subset B_n \cap A$, and each C_i, $1 \leqslant i \leqslant n-2$, is the union of certain (n, i)-segments. The sets C_i are used as a technical device to guarantee that if x and y are in different $(n, i+1)$-segments, then $g(x)$ and $g(y)$ will be in different (n, i)-segments, unless x and $g(x)$, or y and $g(y)$, are in the same (n, i)-segment. The construction goes in stages in such a manner that if $x \in B_n \backslash A$ and $g(x)$ is defined in the ith stage, then x and $g(x)$ lie in the same (n, i)-segment but not in any smaller segment.

We start with the first stage and from the very beginning of the order on B_n. The elements of the sets C_i, $i = -1, 0, 1, ..., n-2$, are called used. Note that only elements of A are unused, and we will show later that there are always unused elements in A at any stage. For $x \in B_n$, if either $x \in A$, or every element in the $(n, 1)$-segment containing x is used, then we do not define $g(x)$ at this stage and move to the next element. Otherwise let $g(x)$ be the first unused element in the order for B_n in the $(n, 1)$-segment containing x, and then add $g(x)$ to C_0. After going through all $x \in B_n$, we go back through the lexicographical order for B_n and start the second stage along the same order. For $x \in B_n$ if $x \notin A$, $g(x)$ is not defined in the previous stage, and there exist unused elements in the $(n, 2)$-segment K containing x, then we define $g(x)$ to be the first unused element in K. We also add $g(x)$ to C_0. If x is the last element in $K \backslash A$ such that $g(x)$ is to be defined (g has been defined for all other elements in $K \backslash A$), then we add the $(n, 1)$-segment containing $g(x)$ into C_1. After that we move to the next element in B_n. Generally, at the jth stage, if $x \notin A$, $g(x)$ is not yet defined in the earlier stages, and there exist unused elements in the (n, j)-segment K containing x, then we define $g(x)$ to be the first unused element in K. We again add $g(x)$ to C_0. For $i = 2, ..., j$ (or $j-1$ if $j = n$), let K_i the the (n, i)-segment containing x. If x is the last element in $K_i \backslash A$ such that $g(x)$ is to be defined, then we add the $(n, i-1)$-segment containing $g(x)$ to C_{i-1}. And then we move to the next element in B_n.

Since for each $(n, i+1)$-segment we add at most one (n, i)-segment to C_i (when the defining for g is finished on it). We see that $|C_i| \leqslant \varepsilon_i \, |B_n|$ for $1 \leqslant i \leqslant n-2$, at any stage. Also $|C_0| \leqslant |C_{-1}| = |B_n \backslash A| \leqslant (1-r)|B_n|$. Thus

$$\sum_{i=-1}^{n-2} |C_i| \leqslant \left[2(1-r) + \sum_{i=1}^{n-2} \varepsilon_i \right] |B_n| < |B_n|,$$

since $\sum \varepsilon_i < r - \frac{1}{2}$ and $r > \frac{1}{2}$. This means that $B_n \backslash \bigcup C_i \neq \varnothing$ and hence we can always find unused elements in B_n. So at the nth stage we can finish defining $g(x)$ for all $x \in B_n \backslash A$, since now $g(x)$ for an undefined x is simply the first unused element in $\underline{B_n}$. The mapping g as defined above has the following properties:

(i) g is a $1-1$ mapping from $B_n \backslash A$ into $B_n \cap A$;

(ii) For any (n, i)-segment K, if $g(K \backslash A) \not\subset K$, then $g^{-1}(K \cap A) \subset K \backslash A$;

(iii) For any (n, i)-segment K with $i \leqslant n-2$, if $g^{-1}(K \cap A) \not\subset K$, then $g^{-1}(K \cap A) \backslash K$ is contained in one $(n, i+1)$-segment.

Property (i) is obvious from the construction. We prove now Properties (ii) and (iii).

Let K be an (n, i)-segment in B_n. At first we assume that $g(K \backslash A) \not\subset K$. Then there exists $x \in K \backslash A$ such that $g(x)$ is defined at level j with $j > i$. Thus at level i, all elements in $K \cap A$ have become used, therefore none of them can be the image point under g defined after level i. This imples (ii).

Secondly we suppose that $g^{-1}(K \cap A) \not\subset K$. Assume that j is the lowest level at which some $y \in K \cap A$ is defined as the image of $x \notin K$ under g. We may also assume that y is the smallest such element in the order for B_n. Let K_1 be the $(n, i+1)$-segment containing x. At level j there are two possible cases. The first is that we exhausted all unused elements in K before leaving K_1. So $g^{-1}(K \cap A) \backslash K \subset K_1$. The second is that the last $x \in K_1$ to be defined is defined at level j and is such that $g(x) \in K$. Then since $j > i$, by the construction, K is added into C_i at this stage. So again, we have $g^{-1}(K \cap A) \backslash K \subset K_1$.

Now for each $x \in B \backslash A$, we define $g(g(x)) = x$, and keep g fixed on $X \backslash ((B \backslash A) \cup g(B \backslash A))$.

Let $\bar{G}[\bar{G}_m]$ be the group generated by $G[G_m]$ and g. We need to prove that the group $\bar{G} = \bigcup \bar{G}_m$ is locally finite. Fix G_m and take $x \in X$. Suppose first $x \in G_m B_n$ for some n. If $n \leqslant m+1$, then x is contained in $\bigcup_{i=1}^{m+1} G_m B_i$, which is disjoint from the other sets B_i, $i > m+1$, and hence is invariant under both G_m and g. Suppose $n > m+1$. Then $x \in B_n(k_{n-1}, ..., k_{m+1}) = K$. If $g(K \backslash A) \not\subset K$, then by Property (ii), $g(K \cap A) \subset K$. And if K' is an (n, m)-segment intersecting $g(K \backslash A) \backslash K$, then $g(K') \subset K \cup K'$ by Property (iii). Thus the union of K and all the (n, m)-segments which intersect $g(K)$ is invariant under both G_m and g. Now assume that $g(K \backslash A) \subset K$, and let K' be the (n, m)-segment containing x. If $g(K') \not\subset K$, then $g(K \cap A)$ meets another $(n, m+1)$-segment and it is reduced to the previous case. If $g(K') \subset K$, then by (iii), the union of all (n, m)-segments in K with this property ($g(K') \subset K$) forms a set invariant under both G_m and g. Finally, if $x \notin G_m B_n$ for any n, then g is fixed on $G_m x$. Thus we obtain a cover of X by G_m-invariant finite subsets of X, each of which has its cardinality at most

$$\max \left\{ 3\,|G_m| \sum_{i=1}^{m+1} |G_i|,\ 3\,|G_{m+1}|(1 + 3\,|G_m|) \right\}$$

and is g-invariant. By Lemma 2, $\bar{G}_m$ is finite. ∎

COROLLARY. *Under the same conditions as in Theorem 2 there exists a countable locally finite group $G' > G$ such that for any G'-invariant mean μ on $l^\infty(X)$, $\mu(A) \geqslant \frac{1}{2}$.*

Proof. By Theorem 2, there is a G-thick set B containing A and a countable locally finite group $\bar{G}$ containing G such that for any $\bar{G}$-invariant mean μ on $l^\infty(X)$, $\mu(B \backslash A) \leqslant \frac{1}{2}$. Since $g(B) = B$ (g as in the theorem), B is also $\bar{G}$-thick. By the corollary of Theorem 1, there exists a countable locally finite group $G' > \bar{G}$ such that for any G'-invariant mean μ on $l^\infty(X)$, $\mu(X \backslash B) = 0$. Therefore $\mu(X \backslash A) \leqslant \frac{1}{2}$. ∎

Now we are ready to prove our main result.

THEOREM 3. *Assuming the continuum hypothesis, there exists a locally finite group G acting on a countable infinite set X, such that there is only one G-invariant mean on $l^\infty(X)$.*

Proof. Let X be any countable infinite locally finite group. Let $G_0 = X$ acting on itself by left multiplication. Suppose $\mathscr{A}$ is the family of all infinite subsets of X, and write $\mathscr{A} = \{A_\alpha\}$, where α runs through all successive countable ordinals (ordinals which are not limits of smaller ordinals). Let α be a countable ordinal, and suppose that we have defined a countable infinite locally finite group G_β for each $\beta < \alpha$, and $G_\beta < G_\gamma$ if $\beta < \gamma < \alpha$. If α is a limit ordinal, let $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$. If $\alpha = \beta + 1$ for some β, then A_α is defined. If A_α is not $G_\beta(r)$-thick for any $r > \frac{1}{2}$, then let $G_\alpha = G_\beta$. If A_α is $G_\beta(r)$-thick for some $r > \frac{1}{2}$, then as in the above corollary, let G_α be a countable locally finite group containing G_β and such that for any G_α-invariant mean μ on $l^\infty(X)$, $\mu(A_\alpha) \geqslant \frac{1}{2}$.

Let $G = \bigcup \{G_\alpha \mid \alpha < \omega_1\}$. Then G is locally finite and for any $A \subset X$, if A is $G(r)$-thick for some $r > \frac{1}{2}$, then $X \backslash A$ is not $G(r)$-thick for any $r > \frac{1}{2}$.

Suppose there are two different G-invariant means μ and ν on $l^\infty(X)$. Then there exists a set $B \subset X$ such that $\mu(B) > \nu(B)$. We may assume that $\mu(B) > \frac{1}{2}$. By the result of Chou [3] on the range of invariant means, the range of μ on subsets of B is the closed interval $[0, \mu(B)]$. Thus we can find a subset C of B such that $\mu(C)$ is a rational number (say p/q) greater than $\frac{1}{2}$ and $\mu(C) > \nu(C) + 1/n$ for some integer n. Divide C into $2np$ subsets $C_1, C_2, \ldots, C_{2np}$ such that $\mu(C_i) = 1/2nq$, $\forall i$, and write them in the increasing order of $\nu(C_i)$. Let $A = \bigcup_{i=1}^{nq+1} C_i$. Then $\mu(A) = (\frac{1}{2}) + (1/2nq) > \frac{1}{2}$, but $\nu(A) < \mu(A) - 1/2n < \frac{1}{2}$. This is a contradiction. ∎

This result can be stated in a more topological way. Let βX denote the Stone–Čech compactification of X. Then the space $l^\infty(X)$ is isomorphically isometric to $C(\beta X)$, the space of all continuous functions on βX. A mean on $l^\infty(X)$ corresponds to a regular Borel probability measure on βX (cf. [4]).

THEOREM 3′. *Assuming the continuum hypothesis, there exists a locally finite group G of self-homeomorphisms of $\beta\mathbb{N}$, such that there is exactly one G-invariant regular Borel probability measure on $\beta\mathbb{N}$. In particular there is only one minimal closed G-invariant subset is $\beta\mathbb{N}$, and the set is contained in $\beta\mathbb{N}\backslash\mathbb{N}$.*

Proof. The first consequence is just a restatement of Theorem 3. The second follows from the fact that every closed G-invariant subset of $\beta\mathbb{N}$ supports a G-invariant measure by Day's fixed point theorem [7] (see [4]). ▮

Remark. Professor Joe Rosenblatt has suggested the following proof for the fact that if there is more than one G-invariant mean on $l^\infty(X)$, then for any $\varepsilon > 0$, there is a subset A of X such that both A and $X\backslash A$ are $G(1-\varepsilon)$-thick. Let μ be an extreme point of the set of all G-invariant means on $l^\infty(X)$, and ν another G-invariant mean, μ and ν can be considered as G-invariant Borel measures on βX. Then ν is not absolutely continuous with respect to μ. Therefore, there exists a G-invariant probability measure ν_1 on βX such that μ and ν_1 are mutually singular (see, e.g., [17, Sect. 10]). Choose a Borel set $E \subset \beta X$ such that $\mu(E) = \nu_1(\beta X\backslash E) = 1$. By the regularity of μ and ν_1 we can find compact sets $F_1 \subset E$ and $F_2 \subset \beta X\backslash E$ such that $\mu(F_1) > 1-\varepsilon$ and $\nu_1(F_2) > 1-\varepsilon$. Let $A \subset X$ be such that $F_1 \subset \bar{A} \subset \beta X\backslash F_2$, where $\bar{A}$ is the closure of A in βX. Then $\mu(A) > 1-\varepsilon$ and $\nu_1(A) < \varepsilon$.

3. EXAMPLES

When $|G| \leqslant |X|$, we know that there are many ($2^{2^{|X|}}$ in fact) G-invariant means on $l^\infty(X)$. Even in the case $|G| > |X|$, there are examples for the nonuniqueness of G-invariant means. For convenience we suppose X is a countable infinite set and G is an uncountable amenable group acting on X.

EXAMPLE 1. G is not transitive. In this trivial case there are two disjoint G-invariant subsets of X, each of which supports a G-invariant mean.

EXAMPLE 2. G is abelian. In this case with our assumption that G is uncountable, we always have that G is not transitive. In fact we have the following more general result. Suppose G acts on X transitively and faithfully, i.e., if $gx = x$ for every $x \in X$, then $g = e$. Fix an element x in X, and let H be the isotropy group of x in G, i.e., $H = \{g \in G \mid gx = x\}$. Then the index $[G:H]$ of H in G is countable, and hence $|H| = |G|$. Take $h \in H$

and $y \in X$ with $hy \neq y$. Choose $g \in G$ such that $gx = y$. Then $(g^{-1}hg)x = g^{-1}(hy) \neq x$. So $g^{-1}hg \notin H$. Thus H does not contain any nontrivial normal subgroup of G. In this case, of course, G cannot be abelian.

EXAMPLE 3. Now we give a serious example. We are going to construct a nilpotent group G and a subgroup H of G, with $|G| = |H| = c$ and $[G:H]$ countable, such that H does not contain any nontrivial normal subgroup of G. Then G acts transitively and faithfully on the set G/H of all left cosets of H in G. By Rosenblatt and Talagrand [19], there is more than one G-invariant mean on $l^\infty(G/H)$.

Let $A = \mathbb{Z}$, and B the weak direct product of countably many $\mathbb{Z}$'s,

$$B = \left\{ \sum_{-\infty}^{\infty} m_i a_i \mid m_i \in \mathbb{Z},\ a_i \text{ independent generators, and only finitely many } m_i \neq 0 \right\}.$$

Let $C = \mathbb{Z}^{\mathbb{Z}}$ with the pointwise addition, i.e., C is the direct product of countably many $\mathbb{Z}$'s.

Define a homomorphism $\rho : C \to \mathrm{Aut}(A \oplus B)$ by

$$\rho_\alpha\left(n, \sum_{-\infty}^{\infty} m_i a_i\right) = \left(n + \sum_{-\infty}^{\infty} \alpha(i)\, m_i,\ \sum_{-\infty}^{\infty} m_i a_i\right), \qquad \alpha \in C = \mathbb{Z}^{\mathbb{Z}}.$$

Let G be the semidirect product of $A \oplus B$ and C with respect to $\rho : G = (A \oplus B) \times_\rho C$. The multiplication for G is defined by

$$\left(n, \sum_{-\infty}^{\infty} m_i a_i, \alpha\right)\left(n', \sum_{-\infty}^{\infty} m'_i a_i, \alpha'\right) = \left(n + n' + \sum_{-\infty}^{\infty} \alpha(i)\, m'_i,\ \sum_{-\infty}^{\infty} (m_i + m'_i)\, a_i,\ \alpha + \alpha'\right).$$

Let Z be the center of G. Then it is easy to see that $Z = \{(n, 0, 0) \mid n \in \mathbb{Z}\}$ and $G/Z \cong B \oplus C$, which is abelian. Thus G is nilpotent of class 2.

Let $H = \{(0, 0, \alpha) \mid \alpha \in C\}$. Then $[G:H]$ is countable, and $|H| = c$. But the only normal subgroup of G contained in H is $\{e\}$. For take any $\alpha \in C$ with $\alpha \neq 0$, say $\alpha(i) \neq 0$. Then

$$(0, a_i, 0)^{-1}\,(0, 0, \alpha)(0, \alpha_i, 0) = (\alpha(i), 0, \alpha) \notin H.$$

Thus $\bigcap_{g \in G} g^{-1}Hg = \{(0, 0, 0)\}$.

Remark. Generally, for an infinite group and its subgroup H, Scott [20] proved that if $|H| > 2^{|G:H|}$, then H contains a nontrivial normal subgroup of G, namely the intersection $\bigcap_{g \in G} g^{-1}Hg$. If $|H| \leqslant 2^{|G:H|}$ and G is abelian, then H itself is normal in G. Our example shows that we cannot go much further from the extreme case of abelian groups.

We conclude with an example which shows that there exist many G-invariant means on X and all of them are supported on the same G-thick sets.

Let G_0 be a countable amenable group acting on an infinite set X. The action can be extended to βX, and in this way G_0 acts on βX. Let K be a minimal closed G_0-invariant subset of βX, and let $\mathfrak{F}$ be the filter of all subsets A of X such that $K \subset \bar{A}$.

LEMMA 3. *If A is a subset of X containing two disjoint G_0-thick sets, then there exists $B \in \mathfrak{F}$ such that $A \backslash B$ is G_0-thick.*

Proof. Suppose A_1 and A_2 are disjoint G_0-thick sets contained in A. Each of $\bar{A}_1$ and $\bar{A}_2$ contains a minimal closed G_0-invariant set, say K_1 and K_2. We assume that K_1 is not K. Then there exists $B \in \mathfrak{F}$ such that $\bar{B} \cap K_1 = \emptyset$. Since $\overline{A \backslash B} \supset K_1$ and K_1 supports a G_0-invariant measure on βX, $A \backslash B$ is G_0-thick in X (cf. [4] for details). ∎

THEOREM 4. *Assuming the continuum hypothesis, there exists a locally finite group G acting on a countable infinite set X such that there is an infinite dimensional set of G-invariant means on $l^\infty(X)$ and all of them are supported on the same G-thick sets.*

Proof. Let X be any countable infinite locally finite group. Let G_0 be X acting on itself by left multiplication. Suppose K is a minimal closed G_0-invariant subset of βX and $\mathfrak{F}$ the filter of all sets $A \subset X$ such that $K \subset \bar{A}$. Write $\mathfrak{F} = \{A_a\}$, where α runs through all successive countable ordinals. Let α be a countable ordinal, and suppose that we have defined a countable locally finite group G_β acting on X for each $\beta < \alpha$, satisfying

(i) $G_\beta < G_\gamma$ if $\beta < \gamma < \alpha$;

(ii) K is G_β-invariant for each $\beta < \alpha$;

(iii) Each G_0-invariant mean supported on K is also G_β-invariant for each $\beta < \alpha$.

If α is a limit ordinal, let $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$. Assume now $\alpha = \beta + 1$. Then A_α is G_β-thick by (iii). By the argument at the end of the proof of Theorem 1, A_α contains two disjoint G_β-thick subsets, and Lemma 3 shows that there is $B \in \mathfrak{F}$ such that $A_\alpha \backslash B$ is G_β-thick. By Theorem 1, there is a permutation g_α of X which maps $X \backslash A_\alpha$ into $A_\alpha \backslash B$ and is fixed on B and such that the group generated by G_β and g_α is still locally finite. Let this group be G_α. Since B is fixed under g_α, each G_β-invariant mean on X is also G_α-

invariant. Let G be the union of all G_α, $\alpha < \omega_1$. Then every G_0-invariant mean supported on K is also G-invariant. By Chou's theorem in [4], they form an infinite dimensional set. Finally suppose μ is a G_0-invariant mean not supported on K. Then there is a set $A_\alpha \in \mathfrak{F}$ such that $\mu(A_a) < 1$. Inductively we may find a sequence of ordinals $\alpha = \alpha_1 < \alpha_2 < \alpha_3 < \cdots$, such that $A_{\alpha_n} \supset A_{\alpha_{n+1}}$ and $g_{\alpha_n}(X \backslash A_{\alpha_n}) \subset A_{\alpha_n} \backslash A_{\alpha_{n+1}}$. Thus if ν is a G_{α_n}-invariant mean on X, then $\nu(X \backslash A_\alpha) \leqslant 2^{-1}\nu(X \backslash A_{\alpha_2}) \leqslant \cdots \leqslant 2^{-n}\nu(X \backslash A_{\alpha_{n+1}}) \leqslant 2^{-n}$. And so μ is not a G-invariant mean. This completes the proof. ∎

From the main results of this paper (Theorems 3 and 4) we see that the uniqueness problem of invariant means is much more complicated when $|G| > |X|$. Here are some open problems one can ask.

(1) What can we say about the uniqueness problem given some other set theoretic assumptions, e.g., Martin's axiom and the negation of the continuum hypothesis $(MA + \neg CH)$?

(2) For what groups G is there always more than one G-invariant mean? (Recently Krasa [13] showed that solvable groups are in this category, which improves Rosenblatt and Talagrand [19, Prop. 6].)

(3) Given an infinite set X, for what subsets M of the set of all means on $l^\infty(X)$ is there an amenable group G acting on X such that M is the set of G-invariant means?

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